Probability and Random Processes

ECS 315

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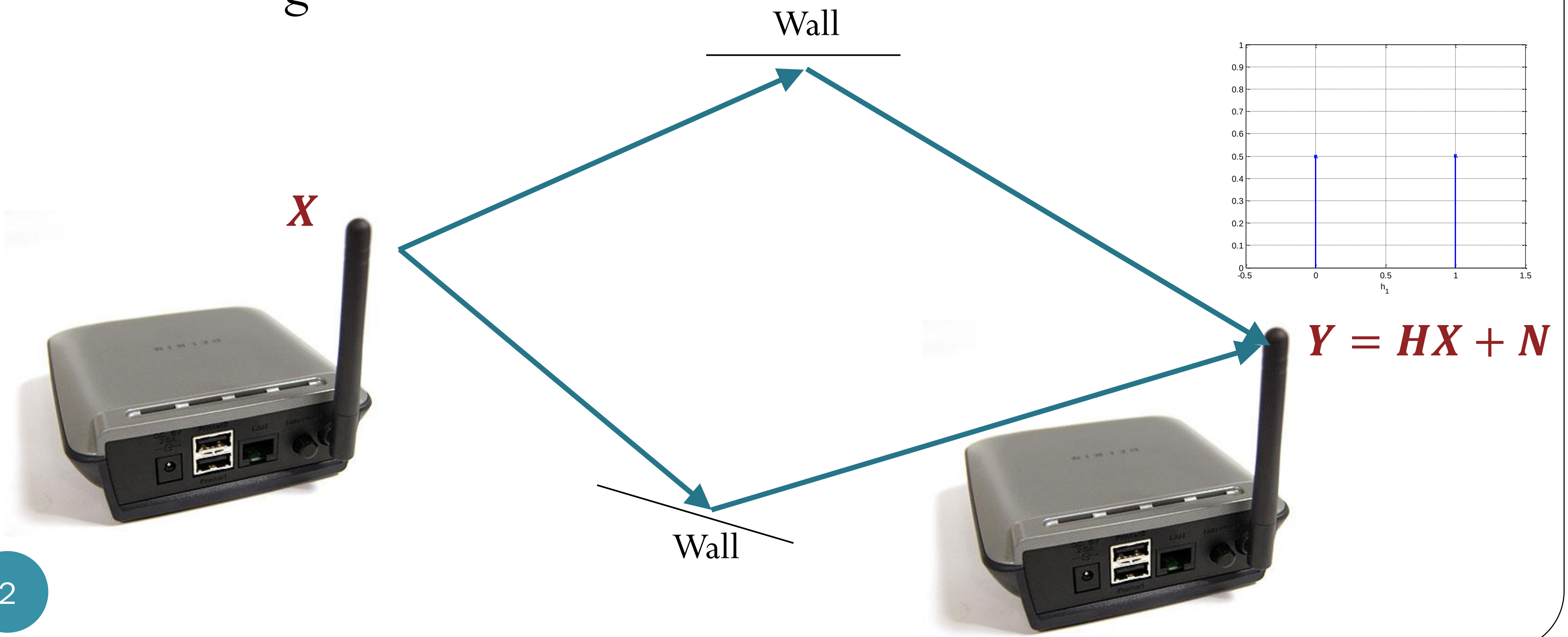
Multiple Random Variables

Office Hours:
BKD 3601-7

Monday	**14:00-16:00**
Wednesday	**14:40-17:00**

SISO Wireless Communications

- Multipath propagation
- At the receiver, multiple copies of the signal may be combined constructively or destructively.
- Fading

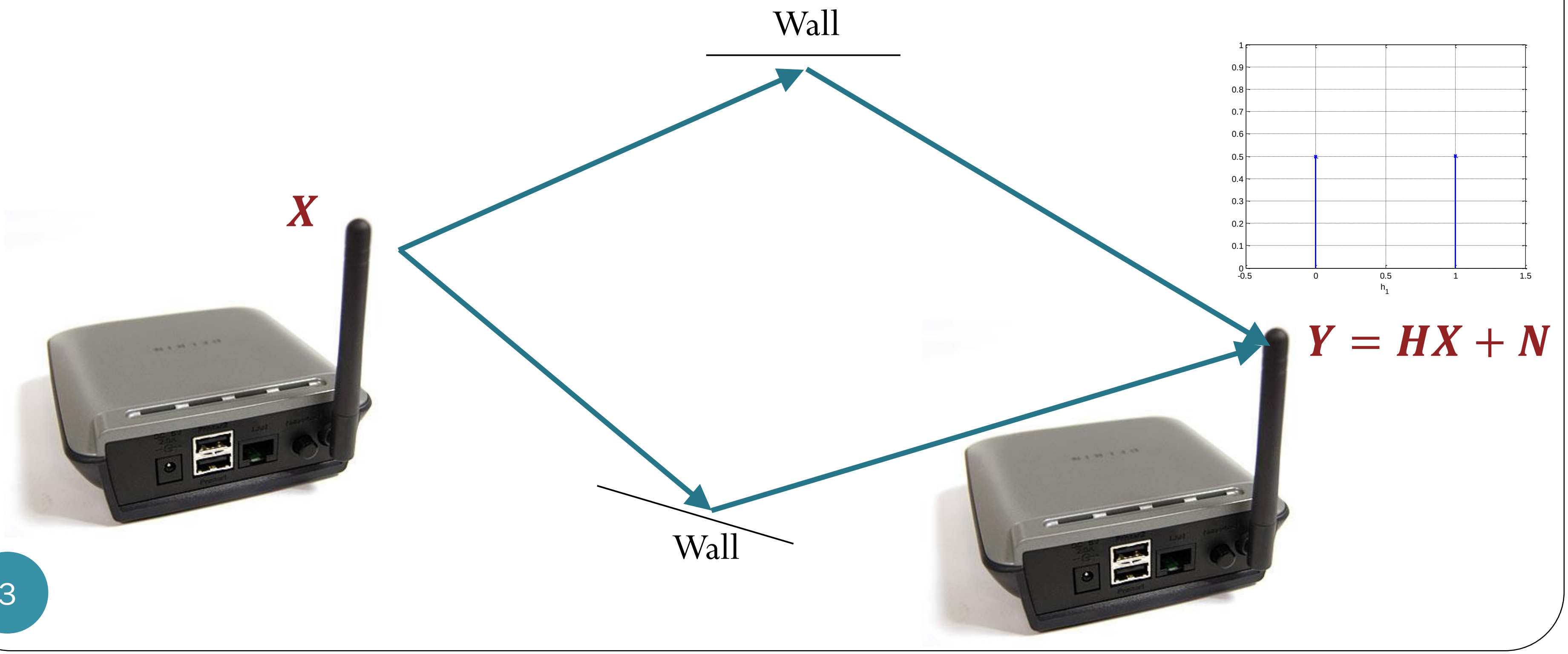


$$Y = HX + N$$

SISO Wireless Communications

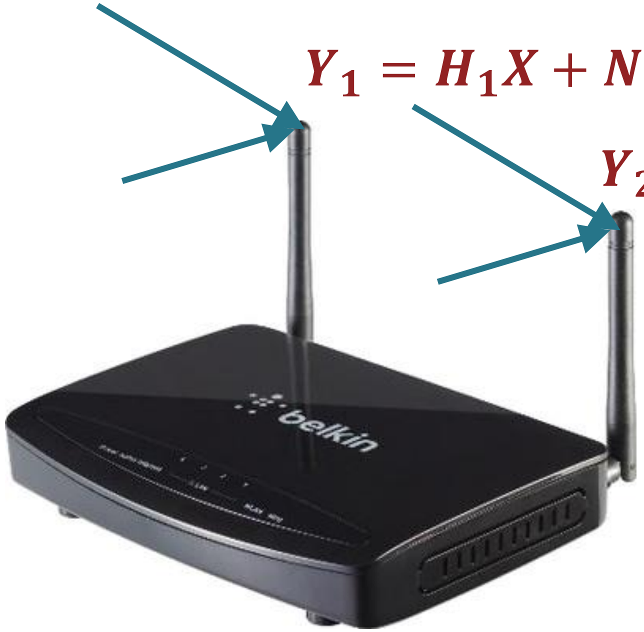
- H = channel coefficient (quality)
- For simplicity, let's assume two possible values for H: good (1) or bad (0).

  Its value is random. H: 0 1 0 0 1 1 1 1 0 1 …



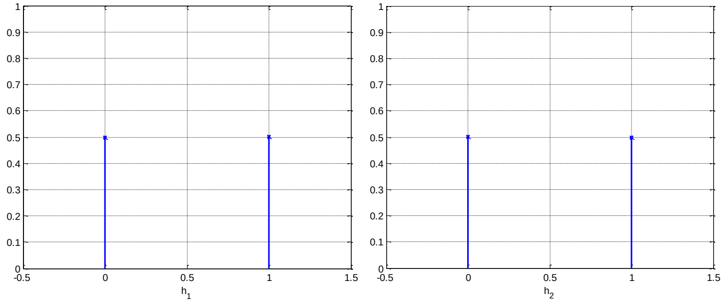
MIMO Wireless Communications

- Here, there are two antennas to receive the signals
- Use the antenna that receive stronger signal (less fading; better channel condition)



$$Y_1 = H_1 X + N_1$$

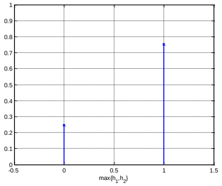
$$Y_2 = H_2 X + N_2$$



H_1: 0 1 0 0 1 1 1 1 0 1 …

H_2: 0 1 1 0 0 0 0 1 0 1 …

H_{used}: 0 1 1 0 1 1 1 1 0 1 …



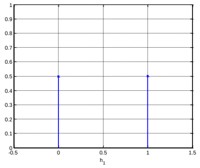
MIMO Wireless Communications

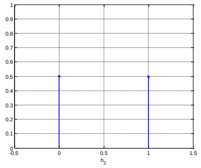
- Here, there are three antennas to receive the signals
- Use the antenna that receive the strongest signal (least fading; best channel condition)

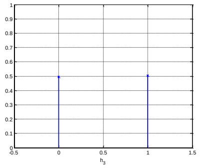
$$\boldsymbol{Y_1 = H_1X + N_1}$$

$$\boldsymbol{Y_2 = H_2X + N_2}$$

$$\boldsymbol{Y_3 = H_3X + N_3}$$





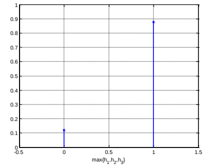


H_1: 0 1 0 0 1 1 1 1 0 1 …

H_2: 0 1 1 0 0 0 0 1 0 1 …

H_3: 0 1 1 1 1 0 1 0 1 1 …

H_{used}: 0 1 1 1 1 1 1 1 1 1 …

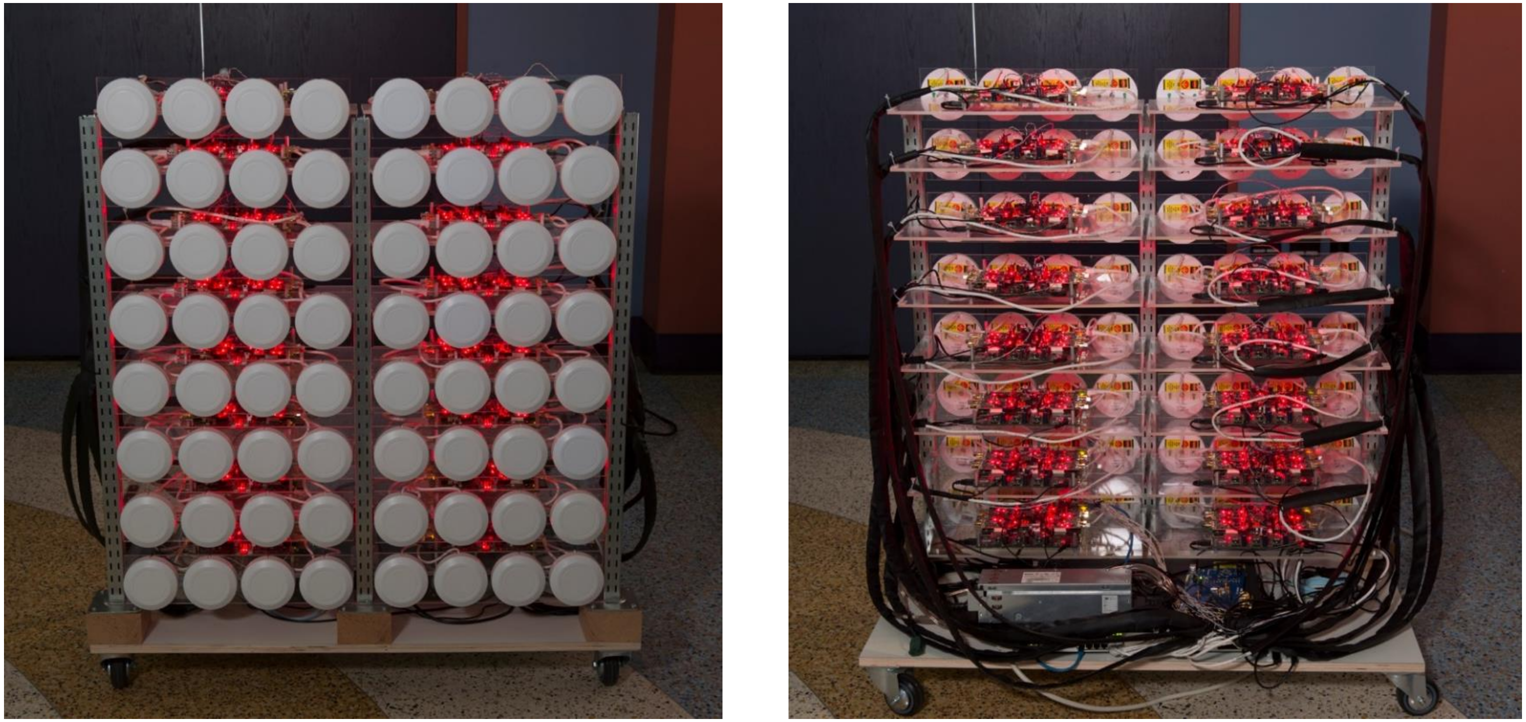


MIMO Communications

- Of course, even more antennas is also possible.

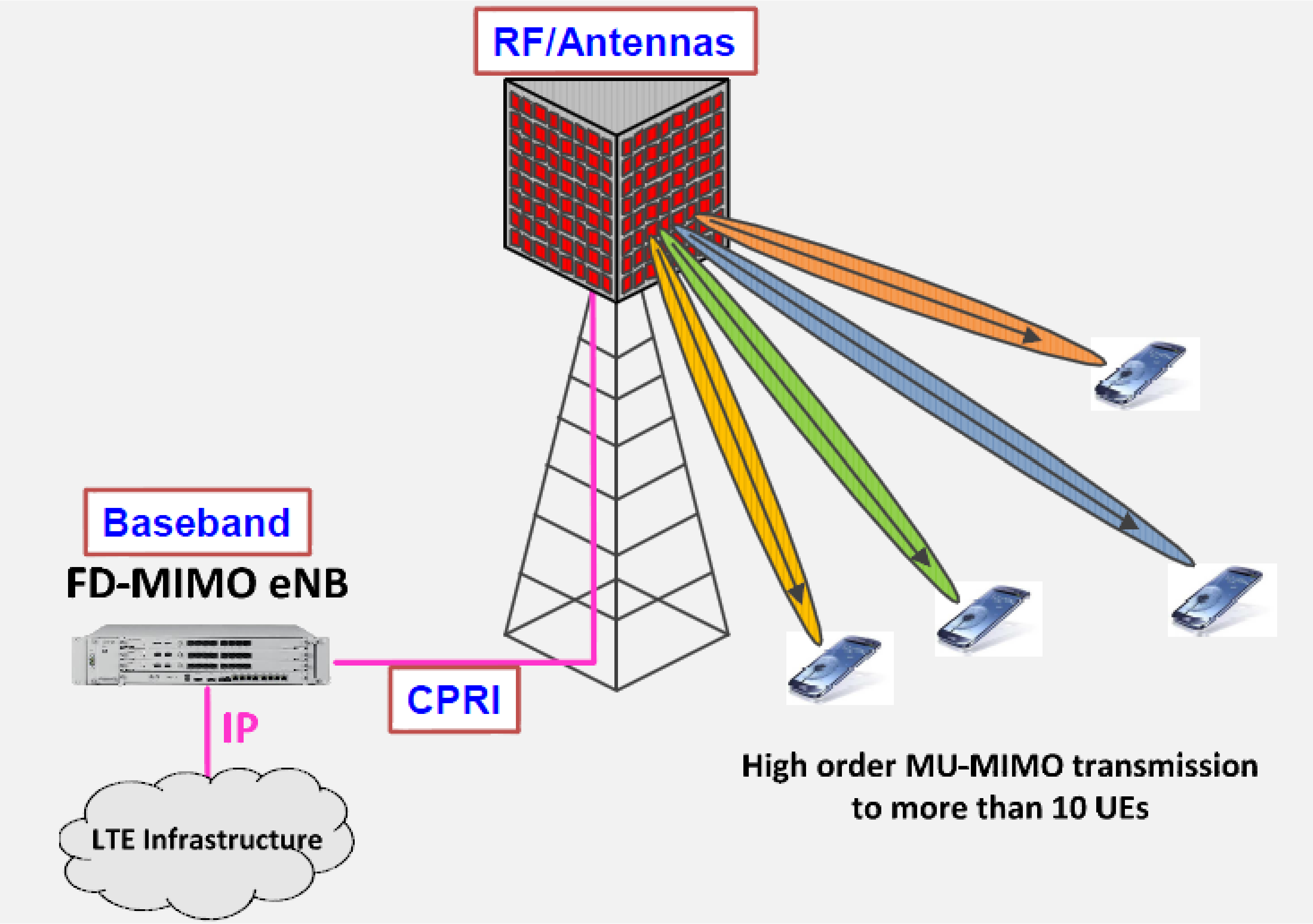
Very Large MIMO Systems

- “antenna array”



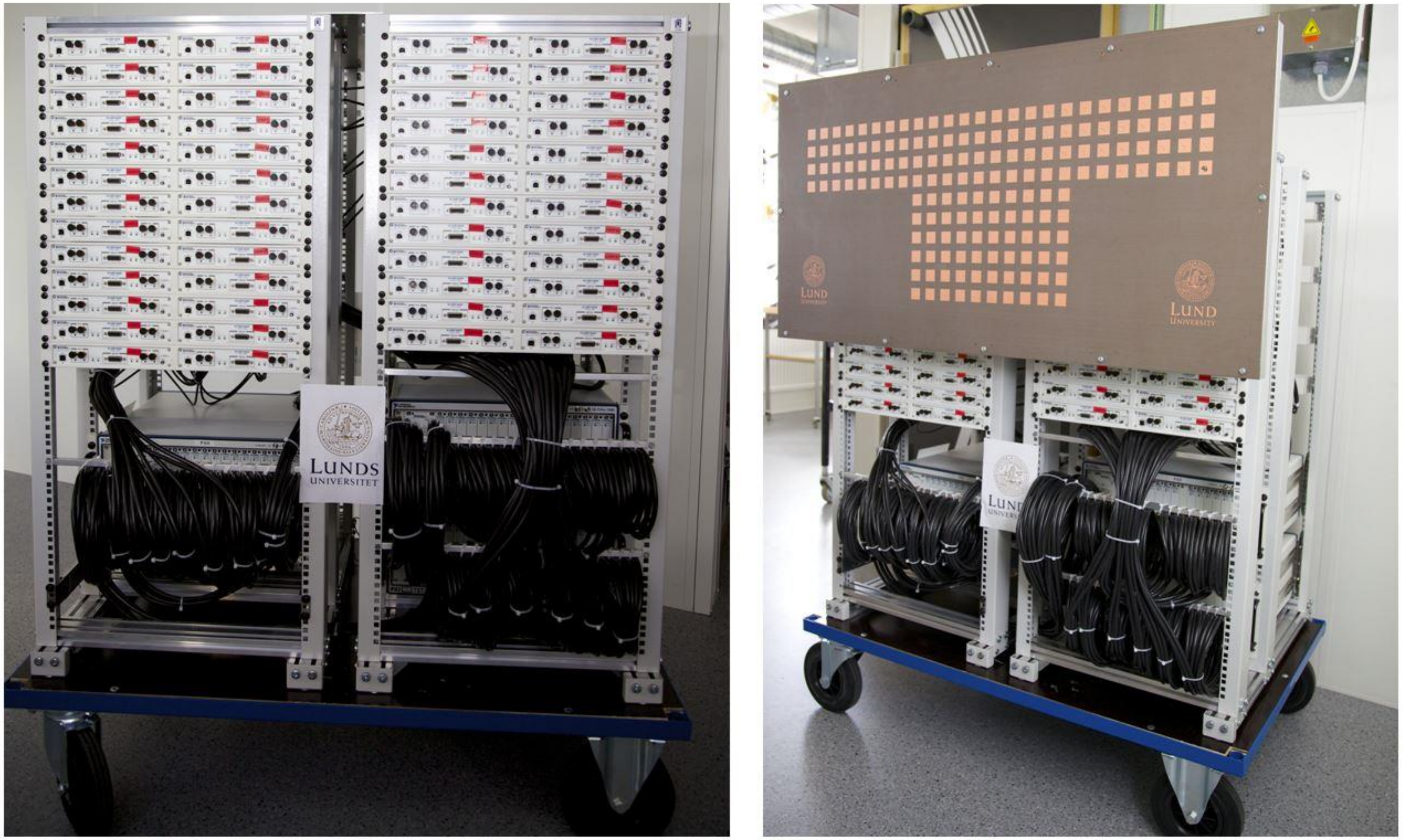
[http://argos.rice.edu/]

Very Large MIMO Systems



[http://www.slideshare.net/zahidtg/full-dimension-mimo-and-beamforming-technologies-for-b4g]

Very Large MIMO Systems



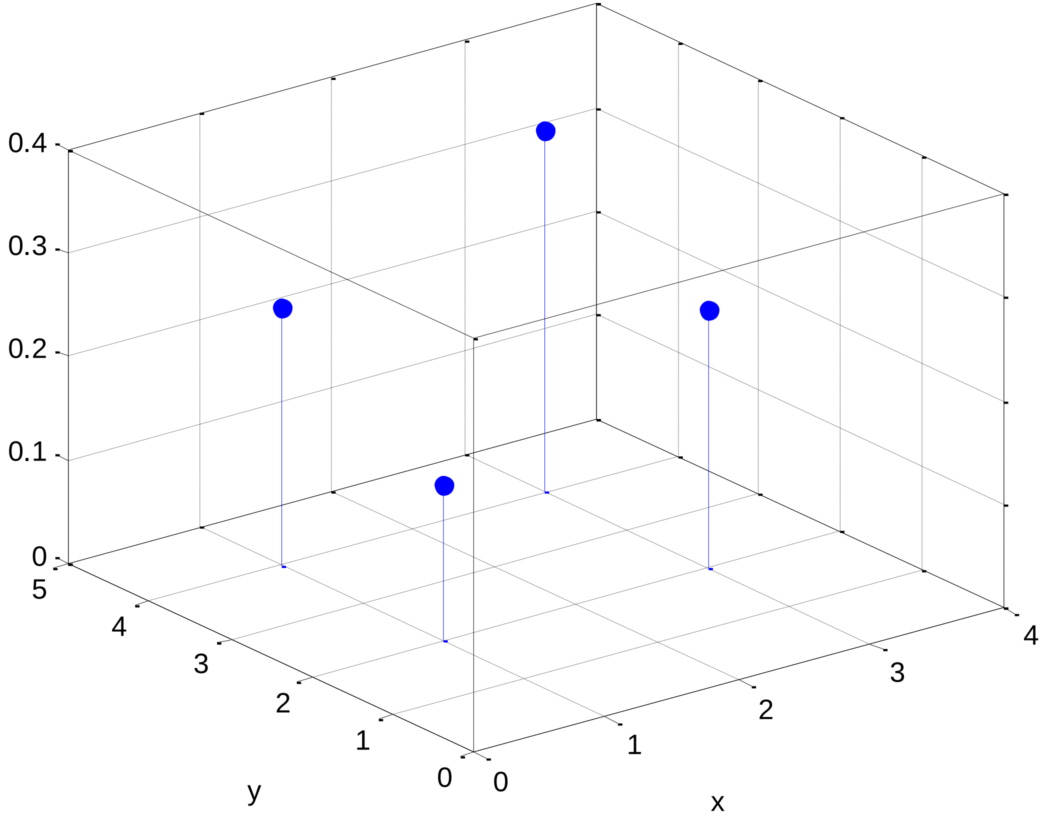
[http://www.ni.com/white-paper/52382/en/]

Very Large MIMO Systems

[http://forums.xilinx.com/t5/Xcell-Daily-Blog/Software-defined-radio-dev-platform-for-5G-research-handles-MIMO/ba-p/441754]

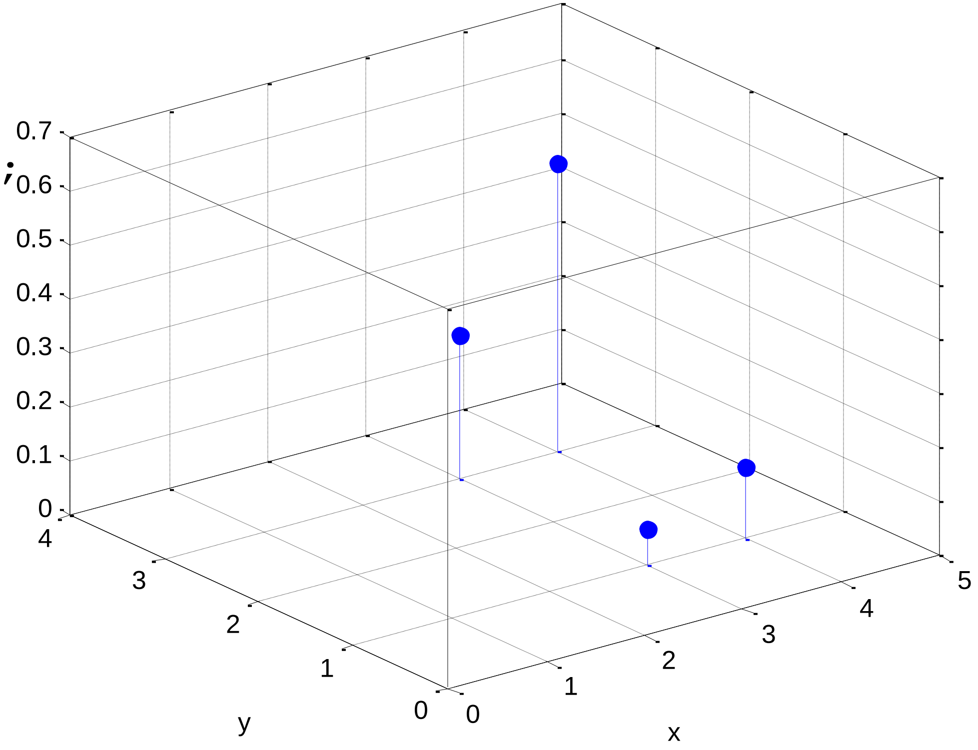
Stem3 (small joint matrix)

```
close all; clear all;
x = [1 3];
y = [2 4];
PXY = [3/20 5/20; 5/20 7/20];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,4])
ylim([0,5])
xlabel('x')
ylabel('y')
```



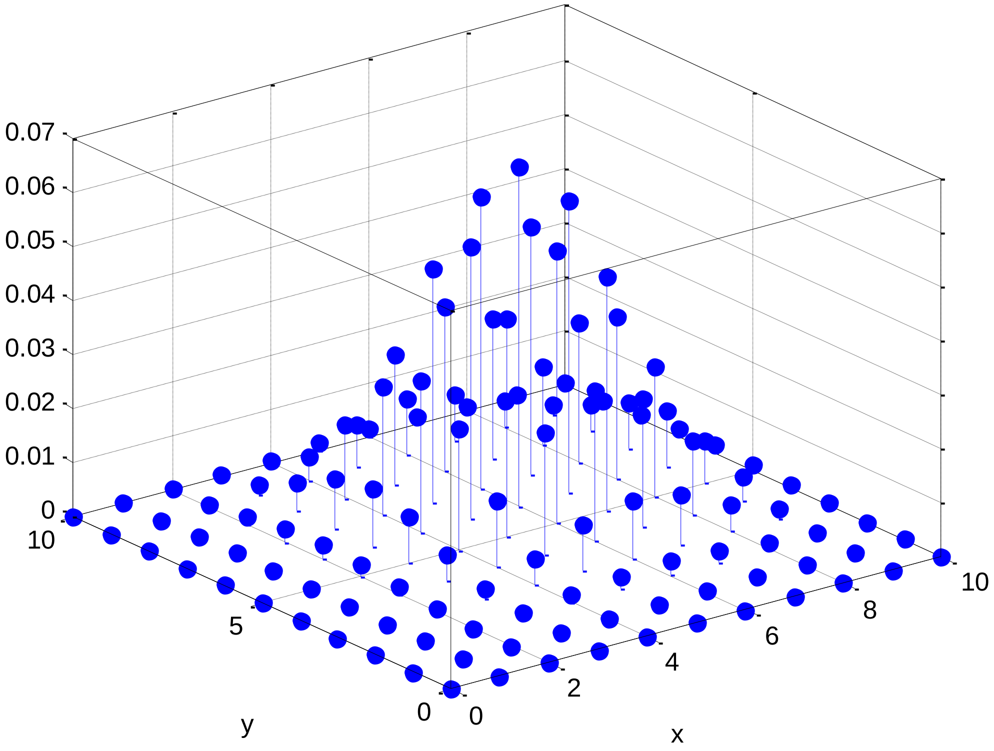
Stem3 (small joint matrix)

```
close all; clear all;
x = [3 4];
y = [1 3];
PXY = [1/15 4/15; 2/15 8/15];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,5])
ylim([0,4])
xlabel('x')
ylabel('y')
```

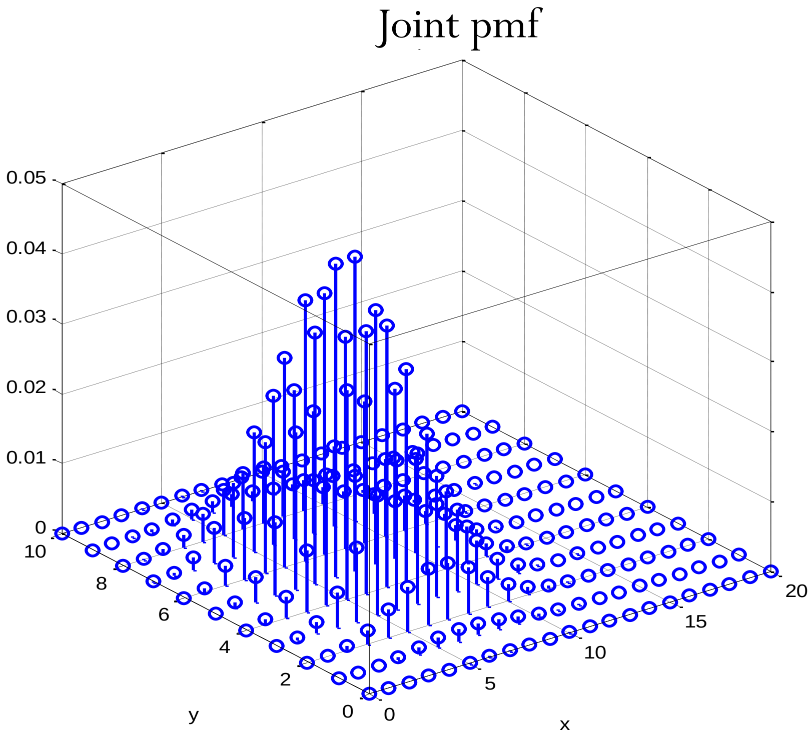


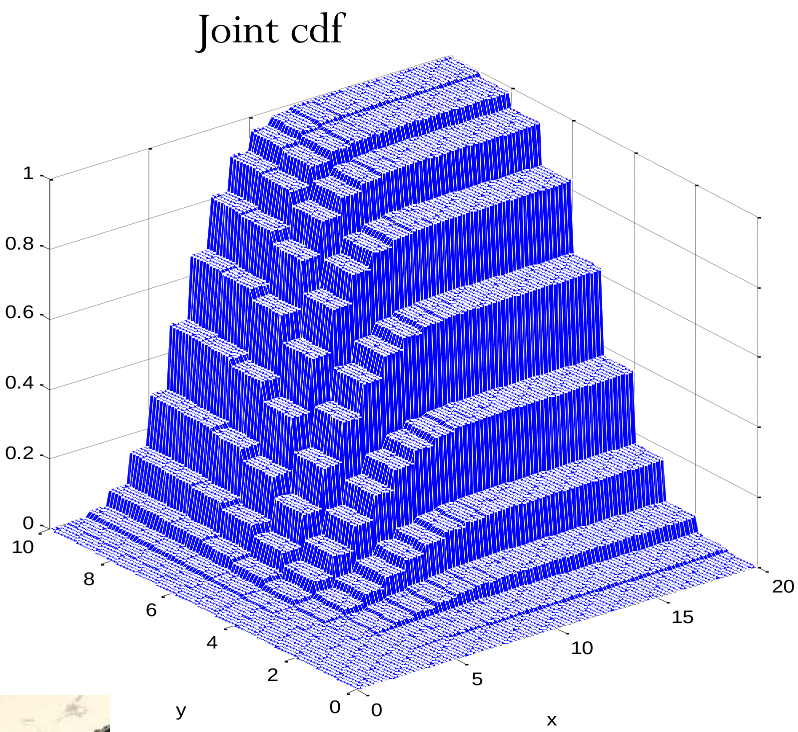
Stem3 (large joint pmf matrix)

```
close all; clear all;
n = 10; p = 3/5;
x = 0:n;
y = 0:n;

pX = binopdf(x,n,p);
pY = binopdf(y,n,p);

PXY = pX.'*pY;

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

%stem3(X,Y,PXY, 'filled')
mesh(X,Y,PXY)
%surf(X,Y,PXY)

xlabel('x')
ylabel('y')
```



Example: Joint pmf and joint cdf

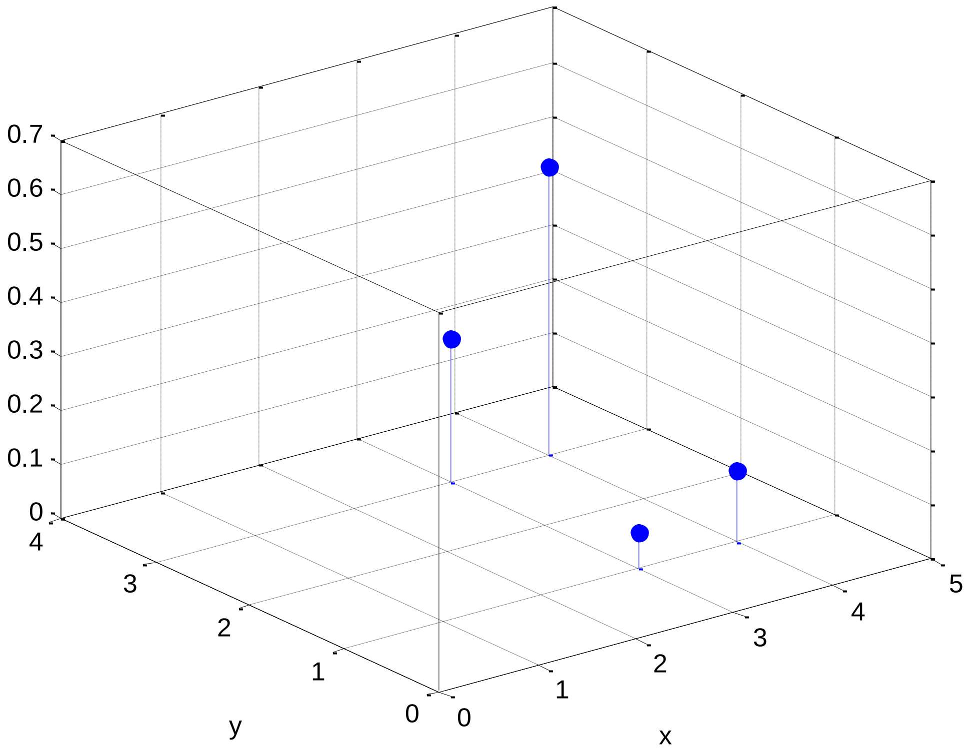




Stem3 (small joint matrix)

$$P_{X,Y} = \begin{array}{cc} {}_{x}\backslash{}^{y} & \begin{array}{cc} 1 & \quad 3 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} {}^{1}/_{15} & {}^{4}/_{15} \\ {}^{2}/_{15} & {}^{8}/_{15} \end{bmatrix} \end{array}$$

```
close all; clear all;
x = [3 4];
y = [1 3];
PXY = [1/15 4/15; 2/15 8/15];

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

stem3(X,Y,PXY,'filled')
xlim([0,5])
ylim([0,4])
xlabel('x')
ylabel('y')
```

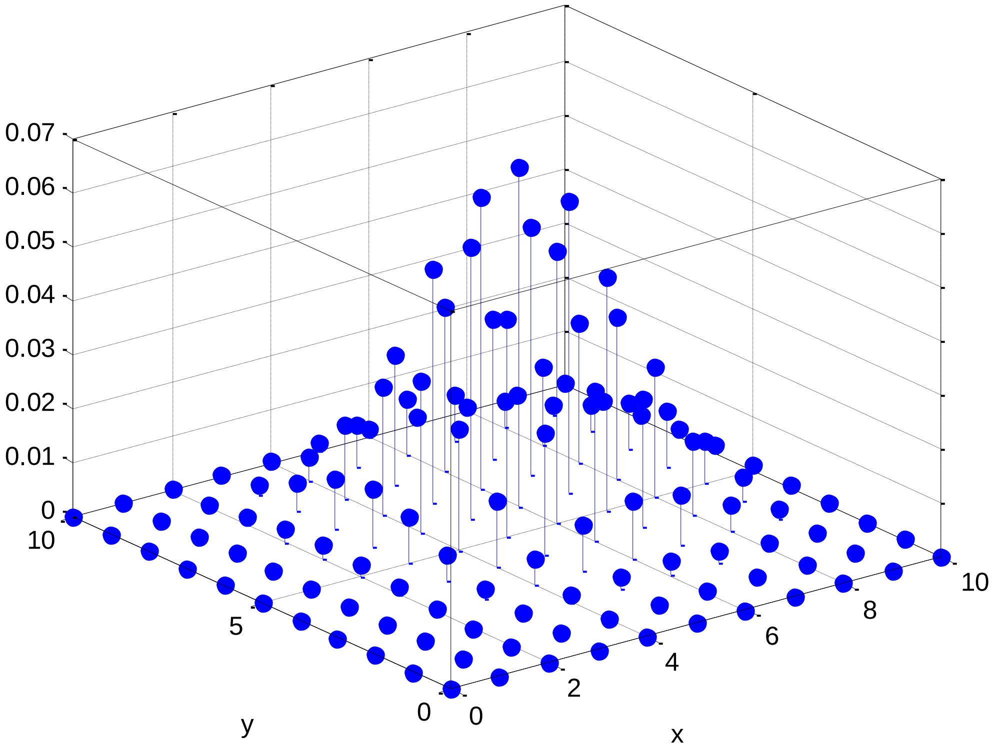


Joint pmf for two i.i.d. RVs

```
close all; clear all;
n = 10; p = 3/5;
x = 0:n;
y = 0:n;

pX = binopdf(x,n,p);
pY = binopdf(y,n,p);

PXY = pX.'*pY;

[X Y] = meshgrid(x,y);
X = X.'; Y = Y.';

%stem3(X,Y,PXY, 'filled')
mesh(X,Y,PXY)
%surf(X,Y,PXY)

xlabel('x')
ylabel('y')
```

Note how the pmfs are multiplied because of the independence.

$$X,Y \overset{\text{i.i.d.}}{\sim} \mathcal{B}\left(10,\frac{3}{5}\right)$$



Evaluation of Probability

- Consider two random variables *X* and *Y*.
- Suppose their **joint pmf matrix** is

$$P_{X,Y} \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[X+Y<7]$

Step 1: Find the pairs (*x*,*y*) that satisfy the condition "x+y < 7"

One way to do this is to first construct the matrix of *x*+*y*.

$$x+y \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 8 & 9 & 10 \\ 6 & 8 & 9 & 10 & 11 & 12 \end{array}$$

Evaluation of Probability

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$P_{X,Y} \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[X + Y < 7]$

Step 2: Add the corresponding probabilities from the joint pmf (matrix)

$$x + y \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 8 & 9 & 10 \\ 6 & 8 & 9 & 10 & 11 & 12 \end{array}$$

$$P[X + Y < 7] = 0.1 + 0.1 + 0.1$$
$$= 0.3$$

Sum of two discrete RVs

- Formula-wise:

$$P[X+Y<7]=\sum_{\substack{(x,y)\\x+y<7}} p_{X,Y}(x,y)$$

Step 2

Step 1

$$x+y \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 8 & 9 & 10 \\ 6 & 8 & 9 & 10 & 11 & 12 \end{array}$$

For our example, only

$$(x,y)\in\left\{\begin{matrix}(1,2),(1,3),(1,4),\\(1,5),(3,2),(3,3),\\(4,2)\end{matrix}\right\}$$

satisfy the condition

- Alternative way to write this:

$$P[X+Y<7]=\sum_{x}\sum_{\substack{y\\x+y<7}} p_{X,Y}(x,y)=\sum_{y}\sum_{\substack{x\\x+y<7}} p_{X,Y}(x,y)$$

Sum of two discrete RVs

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$$P_{X,Y} \quad \begin{array}{c|ccccc} {}_{x}\backslash^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0.1 & 0.1 & 0 & 0 & 0 \\ 3 & 0.1 & 0 & 0 & 0.1 & 0 \\ 4 & 0 & 0.1 & 0.2 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0.3 \end{array}$$

- Find $P[X+Y=7]$

$$P[X+Y=7] = 0.1$$

$$x+y \quad \begin{array}{c|ccccc} {}_{x}\backslash^{y} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 8 & 9 & 10 \\ 6 & 8 & 9 & 10 & 11 & 12 \end{array}$$

Sum of two discrete RVs

- Formula-wise:

Step 2

$$P[X+Y=7]=\sum_{\substack{(x,y)\\x+y=7}} p_{X,Y}(x,y)$$

Step 1

For our example, only $(x,y) \in \{(1,6),(3,4),(4,3)\}$ satisfy the condition

$$x+y \quad \begin{array}{c|ccccc} {}_{x}\diagdown^{y} & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 8 & 9 & 10 \\ 6 & 8 & 9 & 10 & 11 & 12 \end{array}$$

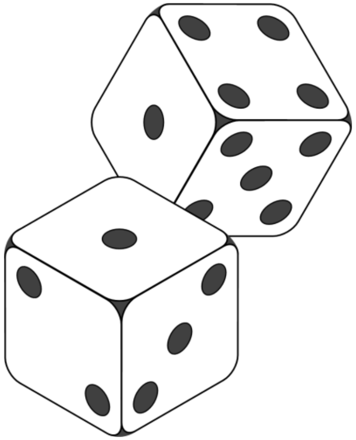
- Other ways to write (and think about) this:

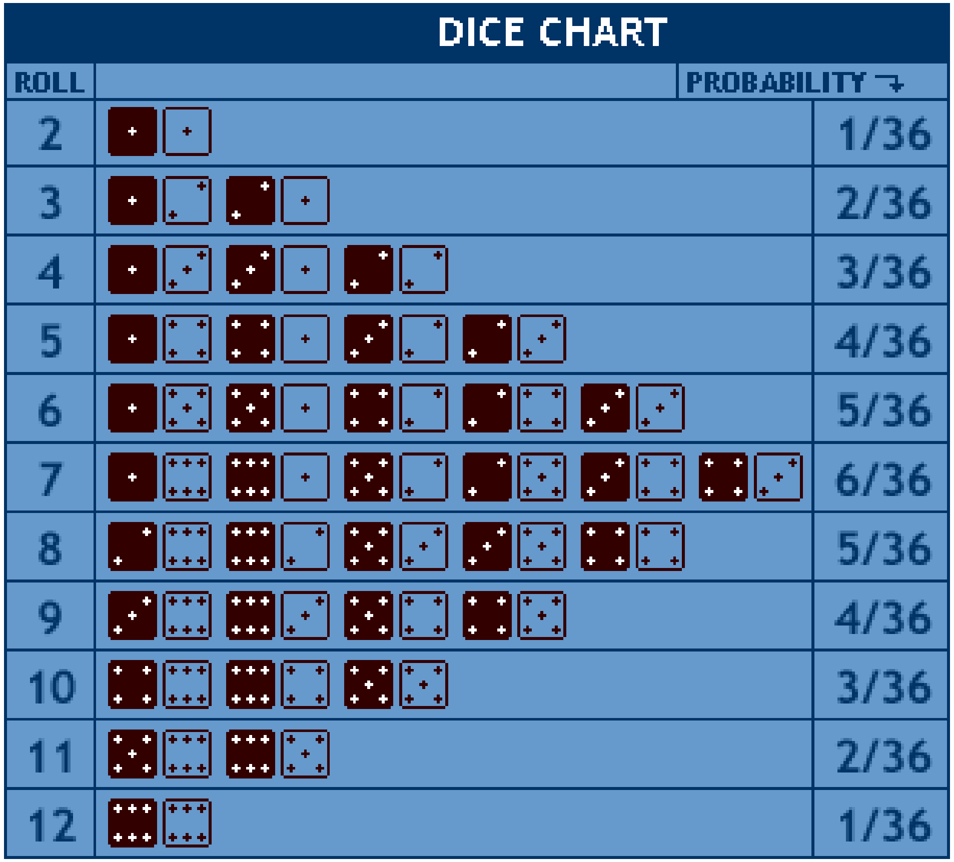
$$P[X+Y=7]=\sum_{x}\sum_{\substack{y\\x+y=7}} p_{X,Y}(x,y)=\sum_{x} p_{X,Y}(x,7-x)$$

$$=\sum_{y}\sum_{\substack{x\\x+y=7}} p_{X,Y}(x,y)=\sum_{y} p_{X,Y}(7-y,y)$$

Sum of Two dice

- Assume that the two dice are fair and independent.





DICE CHART

ROLL		PROBABILITY ↴
2	⚀⚀	1/36
3	⚀⚁ ⚁⚀	2/36
4	⚀⚂ ⚂⚀ ⚁⚁	3/36
5	⚀⚃ ⚃⚀ ⚂⚁ ⚁⚂	4/36
6	⚀⚄ ⚄⚀ ⚃⚁ ⚁⚃ ⚂⚂	5/36
7	⚀⚅ ⚅⚀ ⚄⚁ ⚁⚄ ⚂⚃ ⚃⚂	6/36
8	⚁⚅ ⚅⚁ ⚄⚂ ⚂⚄ ⚃⚃	5/36
9	⚂⚅ ⚅⚂ ⚄⚃ ⚃⚄	4/36
10	⚃⚅ ⚅⚃ ⚄⚄	3/36
11	⚄⚅ ⚅⚄	2/36
12	⚅⚅	1/36

Sum of two indp random variables

- = convolution of their pmf

```
>> pX = ones(1,6)/6
pX =
     0.1667     0.1667     0.1667     0.1667     0.1667     0.1667
>> pY = pX
pY =
     0.1667     0.1667     0.1667     0.1667     0.1667     0.1667
>> pZ = conv(pX,pY)
pZ =
   Columns 1 through 7
     0.0278     0.0556     0.0833     0.1111     0.1389     0.1667
0.1389
   Columns 8 through 11
     0.1111     0.0833     0.0556     0.0278
>> sym(pZ)
ans =
[ 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18, 1/36]
```

Unfortunately, the `conv` command in MATLAB does not work with symbolic numbers.